Some theorems about mutually recursive domain equations in the category of preordered COFEs

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February 2, 2017

1 Basics

In the following let $\mathfrak{P}$ be the category of preordered COFEs and $\mathfrak{C}$ the category of COFEs. There is the forgetful functor $\mathfrak{U} : \mathfrak{P} \to \mathfrak{C}$ which forgets the preorder.

Definition 1.1.

- A functor $F : \mathfrak{P}^{\mathrm{op}} \times \mathfrak{P}^{\mathrm{op}} \to \mathfrak{P}$ is *locally non-expansive* if for all $X, Y, X', Y' \in \mathfrak{P}$, all $n \in \mathbb{N}$, all $f, g : X \to Y$ and $f', g' : X' \to Y'$ we have

$$f \stackrel{n}{=} g \wedge f' \stackrel{n}{=} g' \Rightarrow F(f, f') \stackrel{n}{=} F(g, g')$$

- A functor $F : \mathfrak{P}^{\mathrm{op}} \times \mathfrak{P}^{\mathrm{op}} \to \mathfrak{P}$ is *locally contractive* if for all $X, Y, X', Y' \in \mathfrak{P}$, all $n \in \mathbb{N}$, all $f, g : X \to Y$ and $f', g' : X' \to Y'$ we have

$$f \stackrel{n}{=} g \wedge f' \stackrel{n}{=} g' \Rightarrow F(f, f') \stackrel{n+1}{=} F(g, g')$$

■

Lemma 1.2. *If F is locally non-expansive then $\blacktriangleright \circ F$ and $F \circ \blacktriangleright \times \blacktriangleright$ are both locally contractive where the latter is the functor that on objects acts as $(X, Y) \mapsto F(\blacktriangleright X, \blacktriangleright Y)$.*

2 Mutually recursive domain equations

Theorem 2.1. *Let $F_1, F_2 : \mathfrak{P}^{op} \times \mathfrak{P}^{op} \to \mathfrak{P}$ be two locally contractive functors such that $F_1(1,1)$ and $F_2(1,1)$ are inhabited. Then there exists a pair of objects $X, Y \in \mathfrak{P}$ and a pair of isomorphisms ξ_X and ξ_Y such that*

$$\xi_X : X \xrightarrow{\cong} F_1(X, Y)$$
$$\xi_Y : Y \xrightarrow{\cong} F_2(X, Y).$$

Proof. Since the category $\mathfrak{P}$ is enriched in the category of complete ultrametric spaces so is $\mathfrak{P} \times \mathfrak{P}$. This latter category also has all limits and colimits inherited pointwise from $\mathfrak{P}$ so the general existence theorem from [BST10] applied to the functor

$$G : \mathfrak{P}^{\mathrm{op}} \times \mathfrak{P}^{\mathrm{op}} \to \mathfrak{P} \times \mathfrak{P}$$
$$G(X, Y) = (F_1(X, Y), F_2(X, Y))$$

gives us the solution as described above. □

Suppose now that the functors F_1 and F_2 in Theorem 2.1 are such that $\mathfrak{U} \circ F_1 = \mathfrak{U} \circ F_2$. Then we have

$$\mathfrak{U}(X) \cong \mathfrak{U}(F_1(X, Y)) = \mathfrak{U}(F_2(X, Y)) \cong \mathfrak{U}(Y)$$

so the two solutions X and Y are isomorphic as COFEs, they only differ (potentially) in the preorder. This leads us to the following corollary.

Corollary 2.2. *Let $F_1, F_2 : \mathfrak{P}^{op} \times \mathfrak{P}^{op} \to \mathfrak{P}$ be two locally contractive functors such that $F_1(1,1)$ and $F_2(1,1)$ are inhabited. Further suppose that $F_1(X,Y)$ and $F_2(X,Y)$ differ only in the preorder, i.e., suppose $\mathfrak{U} \circ F_1 = \mathfrak{U} \circ F_2$.*

Then there exists a COFE $X \in \mathfrak{C}$, two preorders $\leq_1$ and $\leq_2$ making X into preordered COFEs X_1 and X_2, and an isomorphism ξ of COFEs

$$\xi : X \to \mathfrak{U}(F_1(X_1, X_2))$$

Let $\leq_{F_1}$ and $\leq_{F_2}$ be preorders on $F_1(X_1, X_2)$ and $F_2(X_1, X_2)$, respectively.

The isomorphism of COFEs ξ further satisfies the following two properties

$$x \leq_1 y \iff \xi(x) \leq_{F_1} \xi(y)$$
$$x \leq_2 y \iff \xi(x) \leq_{F_2} \xi(y)$$

and thus gives rise to two isomorphisms in $\mathfrak{P}$

$$\xi : X_1 \xrightarrow{\cong} F_1(X_1, X_2)$$
$$\xi : X_2 \xrightarrow{\cong} F_2(X_1, X_2).$$

Proof. From Theorem 2.1 we have two objects $X, Y \in \mathfrak{P}$ and two isomorphisms

$$\xi_X : X \xrightarrow{\cong} F_1(X, Y)$$
$$\xi_Y : Y \xrightarrow{\cong} F_2(X, Y).$$

and from the discussion above we have an isomorphism

$$\zeta = \mathfrak{U}\left(\xi_Y^{-1}\right) \circ \mathfrak{U}(\xi_X) : \mathfrak{U}(X) \xrightarrow{\cong} \mathfrak{U}(Y).$$

Define $X_1 = X$ and let $X_2 \in \mathfrak{P}$ be the preordered COFE with $\mathfrak{U}(X)$ as the underlying COFE and the order $\leq_2$ defined as

$$x \leq_2 y \iff \zeta(x) \leq_Y \zeta(y).$$

Then we have $\zeta : X_2 \xrightarrow{\cong} Y$ *in* $\mathfrak{P}$ and so (recalling $X_1 = X$) we have the following isomorphisms in $\mathfrak{P}$

$$\xi_1 = F_1(\mathrm{id}_{X_1}, \zeta) \circ \xi_X : X_1 \cong F_1(X_1, Y) \cong F_1(X_1, X_2)$$
$$\xi_2 = F_2(\mathrm{id}_{X_1}, \zeta) \circ \xi_Y \circ \zeta : X_2 \cong Y \cong F_2(X_1, Y) \cong F_2(X_1, X_2).$$

as required. By definition we have $\mathfrak{U}(\zeta) = \mathfrak{U}\left(\xi_Y^{-1}\right) \circ \mathfrak{U}(\xi_X)$ and so from the assumption $\mathfrak{U} \circ F_1 = \mathfrak{U} \circ F_2$ we get

$$\mathfrak{U}(\xi_2) = \mathfrak{U}(F_2(\mathrm{id}_{X_1}, \zeta)) \circ \mathfrak{U}(\xi_Y) \circ \mathfrak{U}(\zeta) = \mathfrak{U}(F_1(\mathrm{id}_{X_1}, \zeta)) \circ \mathfrak{U}(\xi_X) = \mathfrak{U}(\xi_1)$$

which concludes the proof. The sought for ξ is $\mathfrak{U}(\xi_1)$. □

References

[BST10] Lars Birkedal, Kristian Støvring, and Jacob Thamsborg. The category-theoretic solution of recursive metric-space equations. *Theoretical Computer Science*, 411(47):4102–4122, 2010.